JOURNAL OF APPROXIMATION THEORY **90**, 255–282 (1997)
ARTICLE NO. AT963080

Quadrature Formulae and Polynomial Inequalities

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Communicated by B. Bojanov

Received August 29, 1995; accepted in revised form August 29, 1996

In this paper we prove several inequalities for polynomials and trigonometric polynomials. They are all obtained as applications of certain quadrature formulae, some of which are proved here for the first time. Such an application of a Gaussian quadrature formula was pointed out by Bojanov in 1986 (see *East. J. Approx.* **1** (1995), 37–46; *J. Approx. Theory* **83** (1995), 175–181). Coincidentally, in the same year, it was shown how an inequality for entire functions of exponential type belonging to $L^2(\mathbb{R})$ could be deduced from a Gaussian quadrature formula for the doubly infinite integral $\int_{-\infty}^{\infty} f(x)\,dx$.

1. INTRODUCTION AND STATEMENT OF RESULTS

For any real $\lambda > -1/2$ let w_λ be the ultraspherical weight function

$$w_\lambda(x) := (1-x^2)^{\lambda-1/2}, \qquad -1<x<1 \tag{1}$$

and denote by $\mathscr{F}_\lambda$ the space of all functions f such that $\int_{-1}^{1} w_\lambda(x)\,|f(x)|^2\,dx$ exists. With the weight w_λ we associate the scalar product

$$(\phi, \psi)_\lambda := \int_{-1}^{1} w_\lambda(x)\,\phi(x)\,\psi(x)\,dx, \tag{2}$$

* The first author was sponsored by the CNRS (France), Hutchinson (Paris), and the University of Pau, Contract 710337.

0021-9045/97 $25.00

which is defined on $\mathscr{F}_\lambda$. There exists a sequence of polynomials called ultraspherical or Gegenbauer polynomials which are orthogonal with respect to the scalar product $(\phi, \psi)_\lambda$. As in [2], we shall use for these polynomials the (unusual) notation and normalization

$$C_n^\lambda(x) := \frac{\Gamma(\lambda+1/2)\,\Gamma(n+2\lambda)}{\Gamma(2\lambda)\,\Gamma(n+\lambda+1/2)}\, P_n^{(\lambda-1/2,\,\lambda-1/2)}(x) \qquad (\lambda > -1/2,\ \lambda \neq 0),$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n with $P_n^{(\alpha,\beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(n+1)\,\Gamma(\alpha+1)$. The definition is usually extended to the case $\lambda = 0$, putting $C_n^0(x) := \lim_{\lambda\to 0}(C_n^\lambda(x)/\lambda)$.

Amongst the important special cases of Gegenbauer polynomials are the nth Chebyshev polynomial of the first kind,

$$T_n(x) := \frac{n}{2}\, C_n^0(x),$$

and the nth Chebyshev polynomial of the second kind, $U_n(x) := C_n^1(x)$. We also need to introduce the polynomials

$$Q_n(x) := (x^2-1)\, U_{n-2}(x), \qquad R_n(x) := (x^2-1)\, T_{n-2}(x).$$

We recall the well known fact that

$$\begin{aligned} &\frac{d}{dx}\, C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}(x) \qquad (\lambda > -1/2,\ \lambda \neq 0), \\ &\frac{d}{dx}\, C_n^0(x) = 2C_{n-1}^1(x), \end{aligned} \tag{3}$$

and that the polynomial C_n^λ has n simple zeros which all lie in $(-1, 1)$. We denote them by $x_{n,1}(\lambda), ..., x_{n,n}(\lambda)$ arranged in increasing order. The derivative $(d/dx)\, C_n^\lambda$ has $n-1$ simple zeros in $(-1, 1)$, which, in view of (3), must be $x_{n-1,1}(\lambda+1), ..., x_{n-1,n-1}(\lambda+1)$. Let us denote by $\mathscr{P}_n$ the set of all polynomials of degree at most n whose coefficients may be *nonreal*, and by $\mathscr{P}_n^\lambda$ the class of all polynomials in $\mathscr{P}_n$ satisfying

$$|p(x)| \leqslant |C_n^\lambda(x)| \tag{4}$$

at the zeros of $(x^2-1)(d/dx)\, C_n^\lambda(x)$.

It was proved by Markoff [12] that if p belongs to $\mathscr{P}_n$ and satisfies

$$|p(x)| \leqslant 1 \qquad \text{for} \quad -1 \leqslant x \leqslant 1,$$

then for $k \in \mathbb{N}$

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant T_n^{(k)}(1). \tag{5}$$

The following remarkable extension of this result was obtained by Duffin and Schaeffer (see [9, Theorem II] or [22, pp. 130–138]):

THEOREM A. *The conclusion* (5) *holds if p belongs to* $\mathscr{P}_n$ *and*

$$\left|p\left(\cos\frac{\nu\pi}{n}\right)\right| \leqslant \left|T_n\left(\cos\frac{\nu\pi}{n}\right)\right| = 1, \qquad \nu = 0, 1, ..., n. \tag{6}$$

In addition, they showed that if E is any closed subset of $[-1, 1]$ which does not contain all of the points $\cos(\nu\pi/n)$, then there is a polynomial $p \in \mathscr{P}_n$ bounded by 1 on E for which (5) is not satisfied.

In 1970, Turán asked the following question:

Problem. Let $p \in \mathscr{P}_n$. How large can

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)|$$

be if the graph of p on $[-1, 1]$ lies in the closed unit disk, i.e., if

$$|p(x)| \leqslant \sqrt{1-x^2} \qquad \text{for} \quad -1 \leqslant x \leqslant 1? \tag{7}$$

The answer to this question turned out to be:

THEOREM B [18, 17]. *Let* $p \in \mathscr{P}_n$, $n \geqslant 2$. *If* (7) *is satified, then for all* $k \in \mathbb{N}$ *we have*

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant Q_n^{(k)}(1). \tag{8}$$

This result was subsequently extended as follows:

THEOREM C [20]. *If* $p \in \mathscr{P}_n$ *and even if* (7) *is satisfied only at the zeros of* $(1-z^2)\, T_{n-1}(z)$, *then* (8) *holds for* $k \geqslant 2$. *The same cannot be said if* $k = 1$ *but that*

$$\max_{-1 \leqslant x \leqslant 1} |p'(x)| \leqslant \frac{2}{\pi}\,(1 + 0(1))\, n \log n,$$

which is (*essentially*) *best possible.*

In [17] it was shown that if x^* is any given point of $[-1, 1]$, then amongst all polynomials p in $\mathscr{P}_n$ satisfying (7) on $[-1, 1]$, the quantity $|p^{(k)}(x^*)|$ is maximized by Q_n if x^* belongs to $n-k+1$ well defined disjoint subintervals of $[-1, 1]$ but is not in the complementary subintervals of $[-1, 1]$. As such it is not clear that in the class of polynomials under consideration the quantity $\int_{-1}^{1} w(x)\,|p(k)(x)|^2\,dx$ would be maximized by Q_n unless w vanishes p.p. in the complementary intervals. For this reason we find the following results of Varma (see [27, 28, or 14]) quite interesting.

THEOREM D. *If* $p \in \mathscr{P}_n$ *and* (7) *holds for* $-1 \leqslant x \leqslant 1$, *then*

$$\int_{-1}^{1} \sqrt{1-x^2}\,|p^{(k)}(x)|^2\,dx \leqslant \int_{-1}^{1} \sqrt{1-x^2}\,|Q_n^{(k)}(x)|^2\,dx \qquad (k=2, 3). \tag{9}$$

Furthermore,

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,|p^{(k)}(x)|^2\,dx \leqslant \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,|Q_n^{(k)}(x)|^2\,dx \qquad (k=1). \tag{10}$$

THEOREM E. *Let* $p \in \mathscr{P}_n$ *such that*

$$|p(x)| \leqslant 1-x^2 \qquad (-1 \leqslant x \leqslant 1). \tag{11}$$

Then, we have

$$\int_{-1}^{1} \sqrt{1-x^2}\,|p^{(k)}(x)|^2\,dx \leqslant \int_{-1}^{1} \sqrt{1-x^2}\,|R_n^{(k)}(x)|^2\,dx \qquad (k=2, 3). \tag{12}$$

Earlier, in analogy with Theorem C, it was proved by Rahman and Watt [21] that if $p \in \mathscr{P}_n$, $n \geqslant 3$, and

$$|p(x)| \leqslant 1-x^2 \quad \text{at the points } \cos(v\pi/(n-2)), \qquad v=0, 1, ..., n-2, \tag{13}$$

then for $k \geqslant 3$,

$$\max_{-1 \leqslant x \leqslant 1} |p^{(k)}(x)| \leqslant R_n^{(k)}(1). \tag{14}$$

Here, we prove

THEOREM 1. *Let* $p \in \mathscr{P}_n$. *Even if* (7) *holds only at the zeros of* $(1-z^2)\,T_{n-1}(z)$, *we have*

$$\int_{-1}^{1} (1-x^2)^{k-5/2}\,|p^{(k)}(x)|^2\,dx \leqslant \int_{-1}^{1} (1-x^2)^{k-5/2}\,|Q_n^{(k)}(x)|^2\,dx \qquad (k \geqslant 2).$$

Theorem 1 says, in particular, that in the case $k=3$ inequality (9) remains true even if (7) holds only at the zeros of $(1-z^2)\,T_{n-1}(z)$. Besides, under this weaker assumption, inequality (10) holds for $k=2$ as well.

THEOREM 2. *Let* $p \in \mathscr{P}_n$, $n \geqslant 3$, *such that* (13) *holds. Then*

$$\int_{-1}^{1} (1-x^2)^{k-7/2}\,|p^{(k)}(x)|^2\,dx \leqslant \int_{-1}^{1} (1-x^2)^{k-7/2}\,|R_n^{(k)}(x)|^2\,dx, \qquad (k \geqslant 3).$$

Theorem 2 says, in particular, that (12) holds for $k=4$ as well and even if (11) holds only at the zeros of $(1-z^2)\,T'_{n-2}(z)$.

We also prove

THEOREM 3. *Let* $\lambda \in (-1/2, 1/2]$. *If* $p \in \mathscr{P}_n^{\lambda}$, *then*

$$\int_{-1}^{1} (1-x^2)^{\lambda+k-5/2}\,|p^{(k)}(x)|^2\,dx$$

$$\leqslant \int_{-1}^{1} (1-x^2)^{\lambda+k-5/2}\left|\frac{d^k}{dx^k}\,C_n^{\lambda}(x)\right|^2 dx, \qquad (k \geqslant 2).$$

This result was proved by Varma [27, Theorem 2] for $\lambda=0$, $k=2$ under the stronger restriction that $\max_{-1 \leqslant x \leqslant 1} |p(x)| \leqslant 1$.

Remark 1. From a result of Bojanov [6, Theorem 1 and Remark 2] it follows that if λ and p are as in Theorem 3, then

$$\int_{-1}^{1} (1-x^2)^{\lambda+k-3/2}\,|p^{(k)}(x)|^2\,dx$$

$$\leqslant \int_{-1}^{1} (1-x^2)^{\lambda+k-3/2}\left|\frac{d^k}{dx^k}\,C_n^{\lambda}(x)\right|^2 dx, \qquad (k \geqslant 1).$$

Next, we prove some inequalities for trigonometric polynomials and mention a few corollaries.

THEOREM 4. *If* s *is a trigonometric polynomial of degree at most* n, *such that*

$$\left|s\left(\frac{\nu\pi}{n}\right)\right| \leqslant 1 \qquad \textit{for} \quad \nu = 0, 1, ..., 2n-1 \quad \textit{and} \quad s'(0)=0,$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|s'(\theta)|^2}{1-\cos\theta}\,d\theta\leqslant n^3, \tag{15}$$

with equality if and only if $s(\theta)\equiv e^{i\gamma}\cos n\theta$, $\gamma\in\mathbb{R}$.

COROLLARY 1. *If* p *is a polynomial of degree at most* n, *such that* $|p(x)|\leqslant 1$ *at the zeros of* $(1-x^2)\,T_n'(x)$, *then*

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,|p'(x)|^2\,dx\leqslant \pi n^2, \tag{16}$$

with equality if and only if $p(x)\equiv e^{i\gamma}T_n(x)$, $\gamma\in\mathbb{R}$.

THEOREM 5. *If* s *is a trigonometric polynomial of degree a most* n, *such that*

$$\left|s\left(\frac{v\pi}{n}\right)\right|\leqslant 1 \qquad \text{for} \quad v=0, 1, ..., 2n-1 \quad \text{and} \quad s'(0)=s'(\pi)=0,$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|s'(\theta)|^2}{1-\cos^2\theta}\,d\theta\leqslant n^3, \tag{17}$$

with equality if and only if $s(\theta)\equiv e^{i\gamma}\cos n\theta$, $\gamma\in\mathbb{R}$.

COROLLARY 2. *Under the hypothesis of Corollary* 1, *we have*

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,|p'(x)|^2\,dx\leqslant \pi n^3, \tag{18}$$

with equality if and only if $p(x)\equiv e^{i\gamma}T_n(x)$, $\gamma\in\mathbb{R}$.

Remark 2. Corollary 2 is the case $\lambda=0$, $k=1$ of Bojanov's inequality in Remrark 1.

The next inequality is not new but its proof is. Our proof might seem unduly long but its interest lies in the fact that it is based on a simple quadrature formula.

THEOREM 6. *If* s *is a trigonometric polynomial of degree at most* n, *then*

$$|s(\zeta)|^2\leqslant\frac{\sinh(2n+1)\eta}{\sinh\eta}\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|s(\xi)|^2\,d\xi \qquad (\eta:=\operatorname{Im}\zeta\neq 0). \tag{19}$$

In (19), *equality holds at a point* $\xi_0 + i\eta_0$ *if and only if*

$$s(\zeta) := c\,\frac{\sin(n+\frac{1}{2})(\zeta - \xi_0 + i\eta_0)}{\sin\frac{1}{2}(\zeta - \xi_0 + i\eta_0)}, \qquad c \in C.$$

2. LEMMAS

First we mention three known inequalities and then prove two new ones. After that we present certain positive quadrature formulae as auxiliary results. Only the last formula is really known explicitly.

2.1. *The Inequalities*

The following lemma is a result of Duffin and Schaeffer [9].

LEMMA 2.1. *Let g be a polynomial of degree n whose zeros* $\tau_1, ..., \tau_n$ *are all real and distinct. Further, for* $k \geqslant 2$, *let* $\tau_1^{(k-1)}, ..., \tau_{n-k+1}^{(k-1)}$ *denote the zeros of* $g^{(k-1)}$. *If f is a polynomial of degree at most n such that*

$$|f'(\tau_\nu)| \leqslant |g'(\tau_\nu)| \qquad \textit{for} \quad \nu = 1, ..., n,$$

then for $k \geqslant 2$ *we have*

$$|f^{(k)}(\tau_\nu^{(k-1)})| \leqslant |g^{(k)}(\tau_\nu^{(k-1)})| \qquad \textit{for} \quad \nu = 1, ..., n-k+1.$$

The following two lemmas are taken from [20, 21].

LEMMA 2.2 [20, Lemma 5]. *Let* $p \in \mathscr{P}_n$ *such that* (7) *is satisfied at the zeros of* $(1-z^2)\, T_{n-1}(z)$. *Then at the zeros of* Q_n *we have*

$$|p'(x)| \leqslant |Q_n'(x)| = \begin{cases} 2(n-1) & \textit{if} \quad x = \pm 1 \\ n-1 & \textit{if} \quad U_{n-2}(x) = 0. \end{cases} \tag{20}$$

If equality is attained at one of the zeros, then it is attained at all of them and $p(x) = \gamma Q_n(x)$ *with* $|\gamma| = 1$.

LEMMA 2.3 [21, Lemma 13]. *Let* $p(x) := (1-x^2)\, q(x)$ *be a polynomial of degree at most n such that* $|q(x)| \leqslant 1$ *at* $\lambda_\nu = \cos(\nu\pi/(n-2))$ $(\nu = 0, 1, ..., n-2)$. *Then, at the critical points of* R_n, *we have*

$$|p''(x)| \leqslant |R_n''(x)|. \tag{21}$$

If equality is attained at one of the zeros, then it is attained at all of them and $p(x) = \gamma R_n(x)$ *with* $|\gamma| = 1$.

We need to estimate $|p'(x_{n,\nu}(\lambda)|$ for $p \in P_n^\lambda$. This is done in

LEMMA 2.4. *Let* $\lambda \in (-1/2, 1/2]$. *If* $p \in \mathscr{P}_n^\lambda$, *then at the zeros of* C_n^λ *we have*

$$|p'(x)| \leqslant \left| \frac{d}{dx} C_n^\lambda(x) \right|. \tag{22}$$

Equality holds only if $p(x) = \gamma C_n^\lambda(x)$ *where* $|\gamma| = 1$.

Proof. Let $\lambda \in (-1/2, 1/2]$ be given and consider

$$\psi(x) := (1-x^2) \frac{d}{dx} C_n^\lambda(x) = c_{n+1} \prod_{\nu=0}^{n} (x - x_{n-1,\nu}(\lambda+1)).$$

By Lagrange interpolation with nodes

$$x_{n-1,0}(\lambda+1) = -1, \qquad x_{n-1,1}(\lambda+1), ..., x_{n-1,n}(\lambda+1) = 1,$$

we obain

$$p(x) = \sum_{\nu=0}^{n} \frac{p(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda+1))} \frac{\psi(x)}{x - x_{n-1,\nu}(\lambda+1)}$$

for any $p \in \mathscr{P}_n$, and therefore for $1 \leqslant \mu \leqslant n$,

$$p'(x_{n,\mu}(\lambda)) = \sum_{\nu=0}^{n} \frac{p(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda+1))} \times \frac{(x_{n,\mu}(\lambda) - x_{n-1,\nu}(\lambda+1))\, \psi'(x_{n,\mu}(\lambda)) - \psi(x_{n,\mu}(\lambda))}{(x_{n,\mu}(\lambda) - x_{n-1,\nu}(\lambda+1))^2}$$

Since (see [26])

$$(1-x^2) \frac{d^2}{dx^2} C_n^\lambda(x) - (2\lambda+1) x \frac{d}{dx} C_n^\lambda(x) + n(n+2\lambda)\, C_n^\lambda(x) \equiv 0$$

we have

$$\begin{aligned} \psi'(x) &= (1-x^2) \frac{d^2}{dx^2} C_n^\lambda(x) - 2x \frac{d}{dx} C_n^\lambda(x) \\ &= (2\lambda-1)\, x \frac{d}{dx} C_n^\lambda(x) - n(n+2\lambda)\, C_n^\lambda(x), \end{aligned} \tag{23}$$

and so if $C_n^\lambda(x)=0$ then

$$\frac{(x-x_{n-1,\nu}(\lambda+1))\,\psi'(x)-\psi(x)}{(d/dx)\,C_n^\lambda(x)}=-1-(2\lambda-1)\,xx_{n-1,\nu}(\lambda+1)+2\lambda x^2. \tag{24}$$

Consequently, at the zero of C_n^λ, we have

$$\frac{p'(x)}{(d/dx)\,C_n^\lambda(x)}=\sum_{\nu=0}^{n}-\frac{p(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda+1))}\times\frac{1+(2\lambda-1)\,xx_{n-1,\nu}(\lambda+1)-2\lambda x^2}{(x-x_{n-1,\nu}(\lambda+1))^2} \tag{25}$$

In particular (taking $p(x)\equiv C_n^\lambda(x)$),

$$1=\sum_{\nu=0}^{n}-\lambda_{n,\nu}\,\frac{1+(2\lambda-1)\,x_{n-1,\nu}(\lambda+1)\,x_{n,\mu}(\lambda)-2\lambda(x_{n,\mu}(\lambda))^2}{(x_{n,\mu}(\lambda)-x_{n-1,\nu}(\lambda+1))^2}$$

where

$$\lambda_{n,\nu}:=\frac{C_n^\lambda(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda+1))}.$$

From (23), we have

$$\psi'(x_{n-1,\nu}(\lambda+1))=\begin{cases}-n(n+2\lambda)\,C_n^\lambda(x_{n-1,\nu}(\lambda+1) & \text{if}\quad 1\leqslant\nu\leqslant n-1\\ -2x_{n-1,\nu}(\lambda+1)(d/dx)\,C_n^\lambda|_{x=x_{n-1,\nu}(\lambda+1)} & \text{if}\quad \nu\text{ is 0 or }n.\end{cases}$$

If $\nu=0$ then $x_{n-1,\nu}(\lambda+1)=-1$ and $(d/dx)\,C_n^\lambda|_{x=x_{n-1,\nu}(\lambda+1)}$, $C_n^\lambda(x_{n-1,\nu}(\lambda+1))$ are of opposite signs, from which it readly follows that $\lambda_{n,0}$ is negative. It is similarly seen that $\lambda_{n,n}<0$. That $\lambda_{n,\nu}<0$ for $1\leqslant\nu\leqslant n-1$ is completely obvious from the above expression for $\psi'(x_{n-1,\nu}(\lambda+1))$. Next we observe that if $\lambda\in(-1/2,1/2]$, then

$$\begin{aligned}1+(2\lambda-1)\,x_{n-1,\nu}(\lambda+1)\,x_{n,\mu}(\lambda)-2\lambda(x_{n,\mu}(\lambda))^2&\\ \geqslant 1-|x_{n,\mu}(\lambda)|^2+(2\lambda-1)(|x_{n,\mu}(\lambda)|-|x_{n,\mu}(\lambda)|^2)&\\ =(1-|x_{n,\mu}(\lambda)|)(1+2\lambda\,|x_{n,\mu}(\lambda)|)&\\ \geqslant(1-|x_{n,\mu}(\lambda)|)^2&\\ >0.&\end{aligned}$$

Hence

$$1=\sum_{\nu=0}^{n}\left|\frac{C_n^\lambda(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda))}\right|$$
$$\times\frac{1+(2\lambda-1)\,x_{n-1,\nu}(\lambda+1)\,x_{n,\mu}(\lambda)-2\lambda(x_{n,\mu}(\lambda))^2}{(x_{n,\mu}(\lambda)-x_{n-1,\nu}(\lambda+1))^2}$$

and now, at the zeros of C_n^λ, (4) and (25) give us

$$\frac{|p'(x)|}{|(d/dx)\,C_n^\lambda(x)|}\leqslant\sum_{\nu=0}^{n}\left|\frac{C_n^\lambda(x_{n-1,\nu}(\lambda+1))}{\psi'(x_{n-1,\nu}(\lambda))}\right|$$
$$\times\frac{1+(2\lambda-1)\,x_{n-1,\nu}(\lambda+1)\,x-2\lambda x^2}{(x-x_{n-1,\nu}(\lambda+1))^2}$$
$$=1.$$

The case of equality is easily discussed.

LEMMA 2.5. *Let $\lambda\in(-1/2,1/2]$. If $p\in\mathscr{P}_n^\lambda$ such that $p(z)$ is real for real z, then for all z outside the open unit disk, we have*

$$|p^{(k)}(z)|\leqslant\left|\frac{d^k}{dz^k}\,C_n^\lambda(z)\right|,\qquad(k\geqslant 0).\tag{26}$$

Equality holds only if $p(x)\equiv\gamma C_n^\lambda(x)$ where $\gamma=\pm1$. Inequality (26) *holds for all $z\in\mathbb{R}\backslash(-1,1)$ even if we drop the restriction that $p(z)\in\mathbb{R}$ for $z\in\mathbb{R}$.*

Proof. Let

$$l_\nu(z):=\prod_{\mu=0,\,\mu\neq\nu}^{n}\frac{z-x_{n-1,\mu}(\lambda+1)}{x_{n-1,\nu}(\lambda+1)-x_{n-1,\mu}(\lambda+1)}$$

be the νth fundamental function of Lagrange interpolation with respect to the nodes

$$x_{n-1,0}(\lambda+1)=-1,\qquad x_{n-1,1}(\lambda+1),\,...,\,x_{n-1,n}(\lambda+1)=1.$$

We may write

$$l_\nu(z):=(-1)^{n-\nu}\prod_{\mu=0,\,\mu\neq\nu}^{n}\frac{z-x_{n-1,\mu}(\lambda+1)}{|x_{n-1,\nu}(\lambda+1)-x_{n-1,\mu}(\lambda+1)|},$$

and so

$$\frac{(-1)^j\, l_j(z)}{(-1)^k\, l_k(z)} = c_{j,k}\, \frac{z - x_{n-1,k}(\lambda+1)}{z - x_{n-1,j}(\lambda+1)},$$

where $c_{j,k}$ is a positive constant. At any point z outside the unit disk the diameter $[-1, 1]$ subtends an angle less than $\pi/2$. So any two of the complex numbers $(-1)^\nu\, l_\nu(z)$ seen as vectors in the complex plane make an angle less than $\pi/2$. Since

$$\begin{aligned} p(z) &= \sum_{\nu=0}^{n} p(x_{n-1,\nu}(\lambda+1))\, l_\nu(z) \\ &= \sum_{\nu=0}^{n} (-1)^\nu\, p(x_{n-1,\nu}(\lambda+1)) \cdot (-1)^\nu\, l_\nu(z) \end{aligned}$$

is a sum of real multiples of the vectors $(-1)^\nu\, l_\nu(z)$ which all lie in a sector of opening less than $\pi/2$, the modulus of $p(z)$, i.e., the magnitude of the (vector) sum, will be largest when the quantities $(-1)^\nu\, p(x_{n-1,\nu}(\lambda+1))$ are all positive (or all negative) and as large in magnitude as possible. This means that p should coincide with C_n^λ at each of the points $x_{n-1,0}(\lambda+1)$, ..., $x_{n-1,n}(\lambda+1)$, or else it should coincide with $-C_n^\lambda$ at each of these points. So we have proved that, for each $p \in \mathscr{P}_n^\lambda$, we have

$$|p(z)| \leqslant |C_n^\lambda(z)| \qquad \text{if} \quad |z| > 1.$$

By continuity $|p(z)| \leqslant |C_n^\lambda(z)|$ for $|z| \geqslant 1$. Now note that $C_n^\lambda(z) \neq 0$ for $|z| \geqslant 1$ and so for $|\alpha| > 1$ the polynomial $p(z) - \alpha C_n^\lambda(z)$ has all its zeros in $|z| < 1$. By the Gauss–Lucas theorem, so does $p^{(k)}(z) - \alpha(d^k/dz^k)\, C_n^\lambda(z)$ for $k = 1, ..., n-1$. In other words, $p^{(k)}(z) - \alpha(d^{(k)}/dz^k)\, C_n^\lambda(z) \neq 0$ for $|z| \geqslant 1$ and $k = 1, ..., n$. This is possible only if $|p^{(k)}(z)| \leqslant |(d^k/dz^k)\, C_n^\lambda(z)|$ for $|z| \geqslant 1$, $k = 1, ..., n$.

Now, let us assume that $p \in \mathscr{P}_n^\lambda$, $\lambda \in (-1/2, 1/2]$, but that $p(z)$ is not necessarily real for all real values of z. We will show that (26) still holds for all $z \in \mathbb{R} \backslash (-1, 1)$ and all $k \geqslant 0$. Suppose, if possible, that $|p^{(k)}(\xi)| > |(d^k/dz^k)\, C_n^\lambda(z)|_{z=\xi}$ for some $\xi \in \mathbb{R} \backslash (-1, 1)$ and some $k \geqslant 0$. Let $p^{(k)}(\xi) = |p^{(k)}(\xi)|\, e^{i\gamma}$. If $p(z) := \sum_{j=0}^n a_j z^j$ then $p_1(z) := \sum_{j=0}^n \operatorname{Re}(e^{-i\gamma} a_j)\, z^j$ satisfies all the conditions that p does and in addition $p_1(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. Hence by the first part of the lemma

$$|p_1^{(k)}(\xi)| \leqslant \left| \frac{d^k}{dz^k}\, C_n^\lambda(z) \right|_{z=\xi} < |p^{(k)}(\xi)| = |\operatorname{Re}(e^{-i\gamma} p^{(k)}(\xi))| = |p_1^{(k)}(\xi)|,$$

which is a contradiction.

Remark 3. The idea of the above proof comes from a paper by Bernstein [4]. It was rediscovered by Erdős [10].

Remark 4. The case $k=n$ of (26) says that if $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ and $C_n^\lambda(z)=\sum_{\nu=0}^{n} b_\nu z^\nu$ then $|a_n| \leqslant |b_n|$.

2.2. *The Quadrature Formulae*

We consider quadrature formulae S_m and their respective remainder terms $E_{m,r}$ of algebraic degree of precision (ADP) $2m-r-1$. Thus, for each functions f integrable on $(-1, 1)$,

$$S_m(f)=\sum_{\nu=1}^{m} \omega_\nu f(\tau_\nu) \tag{27}$$

where

$$-1 \leqslant \tau_1 < \cdots < \tau_m \leqslant 1, \qquad \omega_\nu \in \mathbb{R},$$

and

$$E_{m,r}(f)=\int_{-1}^{1} w(x) f(x)\, dx - S_m(f),$$

with

$$E_{m,r}(x^\nu) \begin{cases} =0, & \text{for} \quad \nu=0, ..., 2m-r-1, \\ \neq 0, & \text{for} \quad \nu=2m-r. \end{cases} \tag{28}$$

Such a quadrature formula is called a positive $(2m-r-1, m, w)$ quadrature formula if all its weights ω_ν are nonnegative; for example, the classical Gaussian quadrature is the unique positive $(2m-1, m, w)$ quadrature. Furthermore, we say that a polynomial $q_{m,r} \in \mathscr{P}_m$ generates a positive $(2m-r-1, m, w)$ quadrature formula if it has m zeros $\tau_1 < \cdots < \tau_m$ in $[-1, 1]$ and the interpolatory quadrature formula based on the nodes τ_μ, $\mu=1, ..., m$, is positive. Since the degree of exactness is $2m-r-1$, it is easy to see that the underlying polynomial $q_{m,r}$ must be orthogonal to $\mathscr{P}_{m-r-1}$ with respect to the weight function w. Then, apart from a multiplicative constant, $q_{m,r}$ must be of the form

$$q_{m,r}(x)=C_\lambda^m+\rho_1 C_{m-1}^\lambda+\cdots+\rho_r C_{m-r}^\lambda, \tag{29}$$

where $\rho_1, ..., \rho_r$ are real constants. Such a polynomial is called a quasi-orthogonal polynomial of degree m and order r.

For the historical development and a number of practical computations we refer the reader to [1, 3, and 26]. Positive $(2m-r-1, m, w)$ quadrature formulae have been characterized completely by Peherstorfer [15, 16]. For an earlier paper on the subject see [13]. Recently, Xu [30] has obtained a simpler characterization of positive quadrature formulae for $r=1, 2, 3$ (also see [23] for a different characterization). The case $r=2$ is particularly important for us.

We mention a simple sufficient condition under which a $(2m-3, m, w)$ quadrature formula has only positive weights in the case when $\rho_1=0$ and ρ_2 depends on m.

LEMMA 2.6. *Let $\{F_m\}$ be a family of quasi-orthogonal polynomials of the form*

$$F_m(x) = C_m^\lambda(x) + \rho_{m,2} C_{m-2}^\lambda(x)$$

with $\rho_{m,2} \leqslant 0$. If for all p belonging to $\mathscr{P}_{2m-3}$ we have

$$\int_{-1}^{1} w(x)\, p(x)\, dx = \sum_{\mu=1}^{m} w_\mu p(x_\mu),$$

where $x_1, ..., x_m$ are the zeros of F_m, which all lie in $(-1, 1)$, then the weights w_μ must be all positive.

This lemma is contained in a theorem of Xu [29]. The way he states his result, it might seem that $\rho_{m,2}$ must be independant of m but an examination of his proof shows that it does not have to be so.

The positive weights are very important for the proofs of our results. One positive quadrature formula of Lobatto type which we shall need is the following:

LEMMA 2.7. *For each integer k $(2 \leqslant k \leqslant n)$, there exist a unique system of $n-k+1$ weights $W_{1,1}^{(k)}, ..., W_{n-k+1,1}^{(k)}$ and a positive number A_k such that for any polynomial f in $\mathscr{P}_{2n-2k+1}$ we have*

$$\int_{-1}^{1} (1-x^2)^{k-5/2} f(x)\, dx = \sum_{\nu=1}^{n-k+1} W_{\nu,1}^{(k)} f(x_\nu^{(k-1)}) + A_k\{f(-1)+f(1)\},$$

where $-1 < x_1^{(k-1)} < \cdots < x_{n-k+1}^{(k-1)} < 1$ are the zeros of $Q_n^{(k-1)}$ and

$$A_k = \frac{2^{2k-4}}{(k-1)(2n^2-4n+k)} \frac{\Gamma(k-1/2)\, \Gamma(k-3/2)\, \Gamma(n-k+2)}{\Gamma(n+k-3)}.$$

The weights $W_{\nu,1}^{(k)}$ are all positive.

Proof. It is easily seen that

$$q_{n-k+1}(x) = \frac{1}{n\, 2^{k-3}\Gamma(k-1)}\, Q_n^{(k-1)}(x)$$

$$= C_{n-k+1}^{k-1}(x) - \frac{n-2}{n}\, C_{n-k-1}^{k-1}(x). \tag{30}$$

The zeros of Q_n are all distinct and lie in $[-1, 1]$. So for $2 \leqslant k \leqslant n$ the polynomial $Q_n^{(k-1)}$ has $n-k+1$ distinct zeros $x_\nu^{(k-1)}$, $\nu = 1, ..., n-k+1$, which all lie in the open interval $(-1, 1)$.

Consider the interpolatory quadrature formula

$$\int_{-1}^{1} (1-x^2)^{k-3/2}\, g(x)\, dx \approx \sum_{\nu=1}^{n-k+1} w_\nu^{(k)} g(x_\nu^{(k-1)}) \tag{31}$$

which is exact in $\mathscr{P}_{n-k}$. In view of (30) we have

$$\int_{-1}^{1} (1-x^2)^{k-3/2}\, Q_n^{(k-1)}(x)\, p(x)\, dx = 0, \qquad \forall p \in \mathscr{P}_{n-k-2},$$

and so, by a standard argument, formula (31) is exact for all polynomials in $\mathscr{P}_{2n-2k-1}$. From (30) we see that $\{q_{n-k+1}\}_{n \geqslant 0}$ is a family of quasi-orthogonal polynomials of order 2 with respect to the weight $(1-x^2)^{k-3/2}$. Therefore by Lemma 2.6 the weights $w_\nu^{(k)}$ in the quadrature formula (31) are all positive. Let f belong to $\mathscr{P}_{2n-2k+1}$ and write

$$f(x) = (1-x^2)\, s_1(x)\, Q_n^{(k-1)}(x) + r_1(x)$$

where $s_1 \in \mathscr{P}_{n-k-2}$ and $r_1 \in \mathscr{P}_{n-k+2}$. By Lagrange interpolation in the points

$$-1, x_1^{(k-1)}, ..., x_{n-k+1}^{(k-1)}, 1$$

we have

$$r_1(x) = r_1(-1)\, L_0(x) + \sum_{\nu=1}^{n-k+1} r_1(x_\nu^{(k-1)})\, L_\nu(x) + r_1(1)\, L_{n-k+2}(x)$$

where $L_0, ..., L_{n-k+2}$ are the fundamental polynomials. Since $L_1, ..., L_{n-k+1}$ all vanish at the points -1, 1, we can write

$$f(x) = (1-x^2)\, s_2(x) + r_1(-1)\, L_0(x) + r_1(1)\, L_{n-k+2}(x)$$

$$= (1-x^2)\, s_2(x) + f(-1)\, L_0(x) + f(1)\, L_{n-k+2}(x) \tag{32}$$

where $s_2 \in \mathscr{P}_{2n-2k-1}$. Hence for $f \in \mathscr{P}_{2n-2k+1}$ we have

$$\int_{-1}^{1} (1-x^2)^{k-5/2} f(x)\, dx = \int_{-1}^{1} (1-x^2)^{k-3/2}\, s_2(x)\, dx + af(-1) + bf(1) \quad (33)$$

where

$$a = \int_{-1}^{1} (1-x^2)^{k-5/2}\, L_0(x)\, dx, \qquad b = \int_{-1}^{1} (1-x^2)^{k-5/2}\, L_{n-k+2}(x)\, dx.$$

By the quadrature formula (31) which is exact for all $f \in \mathscr{P}_{2n-2k-1}$, we have

$$\begin{aligned}
\int_{-1}^{1} (1-x^2)^{k-3/2}\, s_2(x)\, dx &= \sum_{\nu=1}^{n-k+1} w_\nu^{(k)} s_2(x_\nu^{(k-1)}) \\
&= \sum_{\nu=1}^{n-k+1} \frac{w_\nu^{(k)}}{1-\{x_\nu^{(k-1)}\}^2} (1-\{x_\nu^{(k-1)}\}^2)\, s_2(x_\nu^{(k-1)}) \\
&= \sum_{\nu=1}^{n-k+1} W_{\nu,1}^{(k)} f(x_\nu^{(k-1)}),
\end{aligned}$$

where

$$W_{\nu,1}^{(k)} = \frac{w_\nu^{(k)}}{1-\{x_\nu^{(k-1)}\}^2}.$$

Here we have also used (32). Hence (33) implies that

$$\int_{-1}^{1} (1-x^2)^{k-5/2} f(x)\, dx = \sum_{\nu=1}^{n-k+1} W_{\nu,1}^{(k)} f(x_\nu^{(k-1)}) + af(-1) + bf(1). \quad (34)$$

Next we note that $a = b$ for the simple reason that the nodes $x_1^{(k-1)}, ..., x_{n-k+1}^{(k-1)}$ are symmetrical about the origin. Further,

$$\begin{aligned}
b &= \int_{-1}^{1} (1-x^2)^{k-5/2} \frac{(1+x)\, q_{n-k+1}(x)}{2q_{n-k+1}(1)}\, dx \\
&= \frac{1}{2q_{n-k+1}(1)} \left\{ \int_{-1}^{1} (1-x)^{k-5/2}\, (1+x)^{k-3/2} \right. \\
&\qquad \left. \times \left\{ C_{n-k-1}^{k-1}(x) - \frac{n-2}{n}\, C_{n-n-1}^{k-1}(x) \right\} dx \right\}.
\end{aligned}$$

Using [2, p. 263, formula 4]

$$\int_{-1}^{1} (1-x)^{\rho}\,(1+x)^{\beta}\,P_m^{(\alpha,\beta)}(x)\,dx$$

$$=\frac{2^{\beta+\rho+1}\Gamma(\rho+1)\,\Gamma(\beta+m+1)\,\Gamma(\alpha-\rho+m)}{m!\,\Gamma(\alpha-\rho)\,\Gamma(\beta+\rho+m+2)}$$

we obtain

$$\int_{-1}^{1} (1-x)^{k-5/2}\,(1+x)^{k-3/2}\,C_{n-k+1}^{k-1}(x)\,dx$$

$$=\int_{-1}^{1} (1-x)^{k-5/2}\,(1+x)^{k-3/2}\,C_{n-k+1}^{k-1}(x)\,dx$$

$$=2^{2k-3}\,\frac{\Gamma(k-3/2)\,\Gamma(k-1/2)}{\Gamma(2k-2)}$$

which implies that

$$a=b=A_k=\frac{2^{2k-3}}{q_{n-k+1}(1)}\,\frac{\Gamma(k-3/2)\,\Gamma(k-1/2)}{n\Gamma(2k-2)}.$$

But

$$q_{n-k+1}(1)=C_{n-k+1}^{k-1}(1)-\frac{n-2}{n}\,C_{n-k+1}^{k-1}(1)$$

$$=\frac{2(k-1)(2n^2-4n+k)(n+k-4)!}{n(2k-3)!\,(n-k+1)!}$$

and so

$$A_k=\frac{2^{2k-4}}{(k-1)(2n^2-4n+k)}\,\frac{\Gamma(k-1/2)\,\Gamma(k-3/2)\,\Gamma(n-k+2)}{\Gamma(n+k-3)}$$

We also have the following positive quadrature formula of Radau type.

LEMMA 2.8. *For each integer k $(2\leqslant k\leqslant n)$, there exist a unique system of $n-k+1$ weights $W_{1,2}^{(k)}, ..., W_{n-k+1,2}^{(k)}$ and a positive number B_k such that for any polynomial f in $\mathscr{P}_{2n-2k}$ we have*

$$\int_{-1}^{1} (1-x)^{k-5/2}\,(1+x)^{k-3/2} f(x)\,dx=\sum_{\nu=1}^{n-k+1} W_{\nu,2}^{(k)} f(y_\nu^{(k-1)})+B_k f(1),$$

where $-1 < y_1^{(k-1)} < \cdots < y_{n-k+1}^{(k-1)} < 1$ *are the zeros of* $Q_n^{(k-1)}$ *and*

$$B_k = \frac{2^{2k-3}}{(k-1)(2n^2-4n+k)} \frac{\Gamma(k-1/2)\,\Gamma(k-3/2)\,\Gamma(n-k+2)}{\Gamma(n+k-3)}$$

The weights $W_{v,2}^{(k)}$ *are all positive.*

The proof of this lemma requires some slight modifications to that of Lemma 2.7.

Remark 5. Let us mention that quadrature formulae of the form

$$\int_{-1}^{1} (1-x^2)^{k-m-3/2} f(x)\,dx = \sum_{v=1}^{n-k+1} W_{v,k,m,2} f(y_v^{(k-1)}) + \sum_{j=0}^{m-1} (a_{j,k} f^{(j)}(-1) + b_{j,k} f^{(j)}(1)),$$

based on the zeros of $Q_n^{(k-1)}$, which are exact for all polynomials in $\mathscr{P}_{2n-2k+2m-1}$, can also be constructed. In addition, some useful information about the "boundary" weights $a_{j,k}$, $b_{j,k}$, $0 \leqslant j \leqslant m-1$, can be obtained, as was done in [3] and [7] for the so-called "generalized quadrature formulae."

We also need the following positive quadrature formula

LEMMA 2.9. *For each integer* k $(3 \leqslant k \leqslant n)$, *there exist a unique system of* $n-k+1$ *weights* $W_{1,3}^{(k)}, \ldots, W_{n-k+1,3}^{(k)}$ *and a positive number* C_k *such that for any polynomial* f *in* $\mathscr{P}_{2n-2k+1}$ *we have*

$$\int_{-1}^{1} (1-x^2)^{k-7/2} f(x)\,dx = \sum_{v=1}^{n-k+1} W_{v,3}^{(k)} f(z_v^{(k-1)}) + C_k\{f(-1)+f(1)\},$$

where $-1 < z_1^{(k-1)} < \cdots < z_{n-k+1}^{(k-1)} < 1$ *are the zeros of* $R_n^{(k-1)}$, *and*

$$C_k = \frac{3 \cdot 2^{2k-6}}{(k-1)(2n^2-8n+3k)} \frac{\Gamma(k-5/2)\,\Gamma(k-3/2)\,\Gamma(n-k+2)}{\Gamma(n+k-5)}.$$

The weights $W_{v,3}^{(k)}$ *are all positive.*

Proof. Starting with the formula

$$\begin{aligned} h_{n-k+1}(x) &:= \frac{1}{n(n-1)\,2^{k-4}\Gamma(k-2)} R_n^{(k-1)}(x) \\ &= C_{n-k+1}^{k-2}(x) - \frac{(n-4)(n-3)}{n(n-1)} C_{n-kl-1}^{k-2}(x), \end{aligned}$$

the result can be obtained in roughly the same way as that for Lemma 2.7.

Now, we mention the following positive quadrature formulaof the Radau type:

LEMMA 2.10. *For each integer k $(3 \leqslant k \leqslant n)$, there exist a unique system of $n-k+1$ weights $W_{1,4}^{(k)}, ..., W_{n-k+1,4}^{(k)}$ and a positive number D_k such that for any polynomial f in $\mathscr{P}_{2n-2k+1}$ we have*

$$\int_{-1}^{1} (1-x)^{k-7/2} (1+x)^{k-5/2} f(x)\, dx = \sum_{\nu=1}^{n-k+1} W_{\nu,4}^{(k)} f(t_\nu^{(k-1)}) + D_k f(1),$$

where $-1 < t_1^{(k-1)} < \cdots < t_{n-k+1}^{(k-1)} < 1$ are the zeros of $R_n^{(k-1)}$, and

$$D_k = \frac{3 \cdot 2^{2k-5}}{(k-1)(2n^2-8n+3k)} \frac{\Gamma(k-5/2)\, \Gamma(k-3/2)\, \Gamma(n-k+2)}{\Gamma(n+k-5)}.$$

The weights $W_{\nu,4}^{(k)}$ are all positive.

This lemma can be proved in roughly the same way as was Lemma 2.9. Finally, we recall the following quadrature formula.

LEMMA 2.11. *If s is a trigonomeric polynomial of degree at most $2n$, then for all real α, we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} s(\xi)\, d\xi = \frac{1}{2n+1} \sum_{\nu=0}^{2n} s\left(\frac{2\nu\pi}{2n+1} + \alpha\right). \tag{35}$$

3. PROOFS OF THE RESULTS

3.1. *Proof of Theorem* 1.

Let $p \in \mathscr{P}_n$. By Lemma 2.7 we have

$$\int_{-1}^{1} (1-x^2)^{k-5/2} \{p^{(k)}(x)\}^2\, dx$$

$$= \sum_{\nu=1}^{n-k+1} W_{\nu,1}^{(k)} \{p^{(k)}(x_\nu^{(k-1)})\}^2 + A_k \{\{p^{(k)}(-1)\}^2 + \{p^{(k)}(1)\}^2\},$$

because $\{p^{(k)}(x)\}^2 \in \mathscr{P}_{2n-2k} \subset \mathscr{P}_{2n-2k+1}$. Since p satisfies (7) at the zeros of $(1-z^2)\,T_{n-1}(z)$, using Lemmas 2.2 and 2.1 and Theorem C, we obtain

$$\int_{-1}^{1} (1-x^2)^{k-5/2}\,\{p^{(k)}(x)\}^2\,dx$$

$$\leqslant \sum_{v=1}^{n-k+1} W_{v,1}^{(k)}\{Q_n^{(k)}(x_v^{(k-1)})\}^2 + A_k\{\{Q_n^{(k)}(-1)\}^2 + \{Q_n^{(k)}(1)\}^2\}$$

$$= \int_{-1}^{1} (1-x^2)^{k-5/2}\,\{Q_n^{(k)}(x)\}^2\,dx,$$

where we have again used Lemma 2.7.

Remark 6. In the above proof, instead of Lemma 2.7, we could have used the Radau-type quadrature formula given in Lemma 2.8.

3.2. *Proof of Theorem* 2

Let p be a polynomial of degree at most n such that (13) holds. By Lemma 2.1 applied in conjunction with Lemmas 2.3 and 2.9 and the inequality (14), we obtain

$$\int_{-1}^{1} (1-x^2)^{k-7/2}\,\{p^{(k)}(x)\}^2\,dx$$

$$= \sum_{v=1}^{n-k+1} W_{v,3}^{(k)}\{p^{(k)}(z_v^{(k-1)})\}^2 + C_k\{\{p^{(k)}(-1)\}^2 + \{p^{(k)}(1)\}^2\}$$

$$\leqslant \sum_{v=1}^{n-k+1} W_{v,3}^{(k)}\{R_n^{(k)}(z_v^{(k-1)})\}^2 + C_k\{\{R_v^{(k)}(-1)\}^2 + \{R_n^{(k)}(1)\}^2\}$$

$$= \int_{-1}^{1} (1-x^2)^{k-7/2}\,\{R_n^{(k)}(x)\}^2\,dx.$$

Remark 7. For the proof of Theorem 2, we could have used, instead of Lemma 2.9, the positive quadrature formula given in Lemma 2.10.

3.3. *Proof of Theorem* 3

Let us recall the generalized Gauss–Lobatto quadrature formula [25, p. 386]

$$\int_{-1}^{1} (1-x^2)^{\lambda-1/2}\,g(x)\,dx$$

$$\simeq \sum_{j=0}^{k-2} (a_j\,g^{(j)}(-1) + b_j\,g^{(j)}(1)) + \sum_{v=1}^{n-k+1} w_v\,g(x_{n-k+1,v}(\lambda+k-1))$$

which is exact for all $g \in \mathscr{P}_{2n-1}$ and wherein the weights w_ν are all positive. Besides, $a_j = (-1)^j b_j > 0$ for all j. We apply this formula to the polynomial $g(x) := (1-x^2)^{k-2} \{p^{(k)}(x)\}^2$. Since

$$g^{(k-2)}(\pm 1) = (\mp 1)^{k-2} 2^{k-2}(k-2)! \, (p^{(k)}(\pm 1))^2$$

we obtain

$$\begin{aligned}
&\int_{-1}^{1} (1-x^2)^{\lambda+k-5/2} \left| \frac{d^k}{dx^k} p(x) \right|^2 dx \\
&\quad = c_{k-2} \left| \frac{d^k}{dx^k} p(-1) \right|^2 + (-1)^{k-2} d_{k-2} \left| \frac{d^k}{dx^k} p(+1) \right|^2 \\
&\qquad + \sum_{\nu=1}^{n-k+1} w_\nu \left| \frac{d^k}{dx^k} p(x_{n-k+1,\nu}(\lambda+k-1)) \right|^2 \\
&\quad \leqslant c_{k-2} \left| \frac{d^k}{dx^k} C_n^\lambda(-1) \right|^2 + (-1)^{k-2} d_{k-2} \left| \frac{d^k}{dx^k} C_n^\lambda(+1) \right|^2 \\
&\qquad + \sum_{\nu=1}^{n-k+1} w_\nu \left| \frac{d^k}{dx^k} C_n^\lambda(x_{n-k+1,\nu}(\lambda+k-1)) \right|^2 \\
&\quad = \int_{-1}^{1} (1-x^2)^{\lambda+k-5/2} \left| \frac{d^k}{dx^k} C_n^\lambda(x) \right|^2 dx
\end{aligned}$$

with $c_{k-2} = 2^{k-2}(k-2)! \, a_{k-2}$, $d_{k-2} = 2^{k-2}(k-2)! \, b_{k-2}$. Here we have used the positivity of a_{k-2}, $(-1)^{k-2} b_{k-2}$, $w_1, ..., w_{n-k+1}$, and Lemmas 2.4, 2.1, and 2.5.

3.5. *Proof of Theorem* 4

According to the famous interpolation formula of Riesz [22], if t is a trigonometric polynomial of degree at most n, then for any θ,

$$t'(\theta) = \frac{1}{4\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{1}{\sin^2(\theta_k/2)} \, t(\theta + \theta_k), \tag{36}$$

where

$$\theta_k = \frac{2k-1}{2n} \pi \qquad (k = 1, ..., 2n).$$

Applying (36) to the trigonometric polynomial $t(\theta) := \sin n\theta$ and setting $\theta = 0$ we obtain

$$\frac{1}{4n} \sum_{k=1}^{2n} \frac{1}{\sin^2(\theta_k/2)} = n. \tag{37}$$

Now if is a trigonometric polynomial of degree at most n such that

$$\left| t\left(\frac{\nu\pi}{n}\right)\right| \leqslant 1 \qquad \text{for} \quad \nu = 0, 1, ..., 2n-1,$$

then by (36)

$$\left| t'\left(\frac{(2\nu-1)\,\pi}{2n}\right)\right| \leqslant n \qquad \text{for} \quad \nu = 1, ..., 2n. \tag{38}$$

Note that

$$s(\theta) := \frac{|t'(\theta)|^2}{1-\cos\theta} \equiv \frac{t'(\theta)\,\overline{t'(\bar{\theta})}}{1-\cos\theta}$$

is a trigonometric polynomial of degree at most $2n-1$ such that

$$|s(\theta)| \leqslant \frac{n^2}{2\sin^2(\theta/2)} \qquad \text{at} \quad \theta := \theta_\nu = \frac{(2\nu-1)\,\pi}{2n} \qquad \text{for} \quad \nu = 1, ..., 2n. \tag{39}$$

Now we apply the quadrature formula

$$\int_{-\pi}^{\pi} s(\theta)\, d\theta = \frac{\pi}{n} \sum_{\nu=1}^{2n} s\left(\frac{(2\nu-1)\,\pi}{2n}\right),$$

valid for all trigonometric polynomials of degree at most $2n-1$, to obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{|t'(\theta)|^2}{1-\cos\theta}\, d\theta &\leqslant \frac{\pi}{n} \sum_{\nu=1}^{2n} \frac{n^2}{2\sin^2(\theta_\nu/2)} \\ &= 2\pi n^2 \frac{1}{4\pi} \sum_{\nu=1}^{2n} \frac{1}{\sin^2(\theta_\nu/2)} \\ &= 2\pi n^3, \end{aligned} \tag{40}$$

where in the last step we have used (37). This is the same as (15).

Remark 8. The above argument can be easily extended to show that if s is a trigonometric polynomial of degree at mostg n, such that

$$\left|s\left(\frac{\nu\pi}{n}\right)\right| \leqslant 1 \qquad \text{for} \quad \nu = 0, 1, ..., 2n-1$$

and

$$s^{(2k-1)}(0) = 0 \qquad \text{for some} \quad k \in \mathbb{N},$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|s^{(2k-1)}(\theta)|}{1-\cos\theta}\, d\theta \leqslant n^{4k-1}.$$

2.6. *Proof of Corollary* 1

Note that if $p \in \mathscr{P}_n$ then $s(\theta) := p(\cos\theta)$ is a trigonometric polynomial of degree at most n such that $|s(\nu\pi/n)| \leqslant 1$ for $\nu = 0, 1, ..., 2n-1$. Besides, $s'(0) = 0$ since $s'(\theta) = -\sin\theta p'(\cos\theta)$. So (15) holds for s. Now

$$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, |p'(x)|^2\, dx = \frac{1}{2}\int_{-\pi}^{\pi} \frac{|s'(\theta)|^2}{1-\cos\theta}\, d\theta$$
$$\leqslant \pi n^3 = \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, |T_n'(x)|^2\, dx.$$

We omit the proofs of Theorem 5 and Corollary 2 since they are very similar to those of Theorem 4 and Corollary 1, respectively.

Remark 9. Instead of Theorem 5 we can just as easily prove that if s is a trigonometric polynomial of degree at most n, such that

$$|s(\nu\pi/n)| \leqslant 1 \qquad \text{for} \quad \nu = 0, 1, ..., 2n-1$$

and

$$s^{(2k-1)}(0) = s^{(2k-1)}(\pi) = 0 \qquad \text{for some} \quad k \in \mathbb{N},$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|s^{(2k-1)}(\theta)|}{1-\cos^2\theta}\, d\theta \leqslant n^{4k-1}.$$

3.7. *Proof of Theorem* 6

In order to estimate $|s(\zeta)|^2$ at a given point $\xi_0 + i\eta_0$ with $\eta_0 \neq 0$, we may assume $\xi_0 = 0$ since we can consider the trigonometric polynomial $s(\xi_0 + \zeta)$ if necessary. Let us prove that if ε is an arbitrary trigonometric polynomial of degree at most n vanishing at $i\eta_0$, then

$$\int_{-\pi}^{\pi} \left| \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} + \varepsilon(\xi) \right|^2 d\xi \geqslant \int_{-\pi}^{\pi} \left| \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} \right|^2 d\xi \quad (41)$$

with equality only if $\varepsilon(\xi) \equiv 0$. We have

$$\begin{aligned}
&\int_{-\pi}^{\pi} \left| \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} + \varepsilon(\xi) \right|^2 d\xi \\
&\quad = \int_{-\pi}^{\pi} \left| \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} \right|^2 d\xi + \int_{-\pi}^{\pi} |\varepsilon(\xi)|^2 d\xi \\
&\qquad + \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} \overline{\varepsilon(\xi)}\, d\xi + \int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(\xi - i\eta_0)}{\sin \frac{1}{2}(\xi - i\eta_0)} \varepsilon(\xi)\, d\xi. \quad (42)
\end{aligned}$$

Further

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})(\xi + i\eta_0)}{\sin \frac{1}{2}(\xi + i\eta_0)} \overline{\varepsilon(\xi)}\, d\xi \\
&\quad = \left(\cosh\left(n+\frac{1}{2}\right)\eta_0\right) \int_{-\pi}^{\pi} \left(\sin\left(n+\frac{1}{2}\right)\xi\right) \frac{\overline{\varepsilon(\xi)}}{\sin \frac{1}{2}(\xi + i\eta_0)}\, d\xi \\
&\qquad + i\left(\sinh\left(n+\frac{1}{2}\right)\eta_0\right) \int_{-\pi}^{\pi} \left(\cos\left(n+\frac{1}{2}\right)\xi\right) \frac{\overline{\varepsilon(\xi)}}{\sin \frac{1}{2}(\xi + i\eta_0)}\, d\xi. \quad (43)
\end{aligned}$$

Now note that $\overline{\varepsilon(\bar{\zeta})}$ vanishes at $-i\eta_0 \pmod{2\pi}$ and so $\overline{\varepsilon(\bar{\zeta} + i\eta_0)}$ vanishes at $0 \pmod{2\pi}$. It follows that if $\overline{\varepsilon(\bar{\zeta} + i\eta_0)} = \sum_{k=-n}^{k} a_k e^{ik\zeta}$ then $\sum_{k=-n}^{n} a_k = 0$, i.e., $w - 1$ is a factor of the polynomial $w^n \sum_{k=-n}^{n} a_k w^k$. Consequently,

$$\frac{\overline{\varepsilon(\bar{\zeta} + i\eta_0)}}{\sin \frac{1}{2}\zeta} = \sum_{k=-(n-1/2)}^{n-1/2} b_k e^{ik(\zeta/2)},$$

i.e.,

$$\frac{\overline{\varepsilon(\bar{\zeta})}}{\sin \frac{1}{2}(\zeta + i\eta_0)} = \sum_{k=-(n-1/2)}^{n-1/2} b_k e^{-(1/2)\,k\eta_0} e^{i(k/2)\zeta}$$

is an entire function of exponential type $n-\frac{1}{2}$. Since $\sin(n+\frac{1}{2})\zeta$ is of exponential type $n+\frac{1}{2}$, the product $A(\zeta):=(\sin(n+\frac{1}{2})\zeta)(\overline{\varepsilon(\bar{\zeta})}/\sin\frac{1}{2}(\zeta+i\eta_0))$ is entire and of exponential type $2n$. It is also periodic with period 2π. Indeed, $\sin(n+\frac{1}{2})(\zeta+2\pi)=-\sin(n+\frac{1}{2})\zeta$ and

$$\frac{\overline{\varepsilon(\overline{\zeta+2\pi})}}{\sin\frac{1}{2}(\zeta+2\pi+i\eta_0)}=\frac{\overline{\varepsilon(\bar{\zeta})}}{-\sin\frac{1}{2}(\zeta+i\eta_0)}.$$

So by a well-known result [5, Theorem 6.10.1], A is a trigonometric polynomial of degree at most $2n$. From Lemma 2.11 it follows that

$$\int_{-\pi}^{\pi} A(\xi)\,d\xi=\int_{-\pi}^{\pi}\left(\sin\left(n+\frac{1}{2}\right)\xi\right)\frac{\overline{\varepsilon(\xi)}}{\sin\frac{1}{2}(\xi+i\eta_0)}\,d\xi=0.$$

Similarly,

$$\int_{-\pi}^{\pi}\cos\left(n+\frac{1}{2}\right)\xi\,\frac{\overline{\varepsilon(\xi)}}{\sin\frac{1}{2}(\xi+i\eta_0)}\,d\xi=0$$

and so

$$\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})(\xi+i\eta_0)}{\sin\frac{1}{2}(\xi+i\eta_0)}\,\overline{\varepsilon(\xi)}\,d\xi=0. \tag{44}$$

Taking conjugates we get

$$\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})(\xi-i\eta_0)}{\sin\frac{1}{2}(\xi-i\eta_0)}\,\varepsilon(\xi)\,d\xi=0. \tag{45}$$

Using (44) and (45) in (42) we conclude that (41) holds and therein equality holds if and only if $\int_{-\pi}^{\pi}|\varepsilon(\xi)|^2\,d\xi=0$; but ε being continuous, $\int_{-\pi}^{\pi}|\varepsilon(\xi)|^2\,d\xi=0$ if and only if $\varepsilon(\xi)\equiv 0$.

Now let s be an arbitrary trigonometric polynomial of degree at most n with $s(i\eta_0)=\lambda_0$ and consider $t(\zeta):=(1/\lambda_0)(\sinh(2n+1)\eta_0/\sinh\eta_0)\,s(\zeta)$. Since t takes the same value at the point $i\eta_0$ as the trigonometric polynomial $(\sin(n+\frac{1}{2})(\zeta+i\eta_0))/(\sin\frac{1}{2}(\zeta+i\eta_0))$ does, it follows from (41) that

$$\int_{-\pi}^{\pi}\left[\frac{\sin(n+\frac{1}{2})(\xi+i\eta_0)}{\sin\frac{1}{2}(\xi+i\eta_0)}\right|^2 d\xi\leqslant\frac{1}{|\lambda_0|^2}\left(\frac{\sinh(2n+1)\eta_0}{\sinh\eta_0}\right)^2\int_{-\pi}^{\pi}|s(\xi)|^2\,d\xi.$$

It is easily seen that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+\frac{1}{2})(\xi+i\eta_0)}{\sin\frac{1}{2}(\xi+i\eta_0)}\right|^2 d\xi=\frac{\sinh(2n+1)\eta_0}{\sinh\eta_0}$$

and so

$$|s(i\eta_0)|^2=|\lambda_0|^2\leqslant\frac{\sinh(2n+1)\eta_0}{\sinh\eta_0}\int_{-\pi}^{\pi}|s(\xi)|^2\,d\xi,$$

which readly leads us to (19).

Remarks on Theorem 6. (i) Let $|s(\xi_0)|=\max_{\xi\in\mathbb{R}}|s(\xi)|$. Taking $\zeta=\xi_0+i\eta$ in (19) and letting η tend to zero we obtain

$$\max_{\xi\in\mathbb{R}}|s(\xi)|=|s(\xi_0)|\leqslant\sqrt{2n+1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|s(\xi)|^2\,d\xi\right)^{1/2}.$$

(ii) Let $p\in\mathscr{P}_n$. Then $s(\zeta):=p(\cos\zeta)$ is a trigonometric polynomial of degree at most n. So by (19), we have for $\eta\neq 0$,

$$|p(\cos(\xi+i\eta))|^2\leqslant\frac{\sinh(2n+1))\eta}{\sinh\eta}\frac{1}{2\pi}\int_{-\pi}^{\pi}|p(\cos\xi)|^2\,d\xi,$$

i.e., for all $\xi\in\mathbb{R}$,

$$\left|p\left(\frac{e^{\eta}+e^{-\eta}}{2}\cos\xi-i\frac{e^{\eta}-e^{-\eta}}{2}\sin\xi\right)\right|^2$$
$$\leqslant\frac{\sinh(2n+1)\eta}{\sinh\eta}\frac{1}{\pi}\int_{-1}^{1}|p(x)|^2\frac{dx}{\sqrt{1-x^2}}.$$

As ξ varies, $((e^{\eta}+e^{-\eta})/2)\cos\xi-i((e^{\eta}-e^{-\eta})/2)\sin\xi$ describes the ellipse with foci at $-1, 1$ and semi-axes $((e^{|\eta|}+e^{-|\eta|})/2)$, $((e^{|\eta|}-e^{-|\eta|})/2)$. So, putting $R=e^{|\eta|}>1$ and denoting by $\mathscr{E}_R$ the ellipse $x^2/((R+R^{-1})/2))^2+y^2/((R-R^{-1})/2)^2=1$, we conclude that

$$\max_{z\in\mathscr{E}_R}|p(z)|\leqslant\sqrt{\frac{R^{2n+1}-R^{-(2n+1)}}{R-R^{-1}}}\left(\frac{1}{\pi}\int_{-1}^{1}|p(x)|^2\frac{dx}{\sqrt{1-x^2}}\right)^{1/2}.$$

Some Additional Remarks. With reference to Lemma 2.7, the question arises of whether the quadrature formula given therein has ADP $>$ $2n-2k+1$. The answer is no. Indeed, the polynomial

$$p(x) = (1-x^2)\left\{C_{n-k+1}^{k-1}(x) - \frac{n-2}{n}\, C_{n-k-1}^{k-1}(x)\right\} C_{n-k-1}^{k-1}(x)$$

vanishes at the nodes of the quadrature formula, but

$$\int_{-1}^{1} (1-x^2)^{k-5/2}\, p(x)\, dx$$

$$= -\frac{n-2}{n} \int_{-1}^{1} (1-x^2)^{k-3/2} \left\{C_{n-k-1}^{k-1}(x)\right\}^2 dx < 0 \qquad (n > 2).$$

Therefore ADP $= 2n-2k+1$. The ADP of quadrature formulae given in Lemmas 2.8, 2.9, and 2.10 can be similarly discussed.

The nodes of quadrature formulae given in Lemmas 2.7, 2.8, 2.9, and 2.10 are not available in an explicit form, but they are all zeros of certain quasi-orthogonal polynomials. These polynomials can be expressed as characteristic polynomials of a symmetric tridiagonal matrix, as was explained in detail in [30]. Hence, the nodes can be found numerically, by using an appropriate method for the computation of eigenvalues, as was done in [11] for classical orthogonal polynomials.

It may be added that we can also prove quadrature formulae of the form

$$\int_{-1}^{1} (1-x^2)^{k-m-2+\lambda} f(x)\, dx = \sum_{\nu=1}^{n-k+1} W_{\nu,k,m,\lambda}\, f(y_\nu^{(k-1)})$$

$$+ \sum_{i=0}^{m-1} \left(a_{i,k,m,\lambda}\, f^{(j)}(-1) + b_{j,k,m,\lambda}\, f^{(j)}(1)\right),$$

based on the zeros of $(d^{(k-1)}/dx^{(k-1)})\{(x^2-1)(d/dx)\, C_{n-2}^{\lambda}\}$, and which are exact for all polynomials in $\mathscr{P}_{2n-2k+2m-1}$.

In future work we will report on new positive quadrature formulae based on the zeros of the derivatives of certain quasi-orthogonal polynomials, which satisfy certain boundary conditions. This type of quadrature formulae are specially interesting, since they can be used to introduce and justify new collocation methods for the numerical solution of partial differential equations. We refer the reader to [3] for a review of these methods.

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